Conditions for eigenvalue configurations of two real symmetric matrices

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Problem

Definition (Eigenvalue configuration)
Let $F \in \mathbb{R}^{m\times m}$ and $G \in \mathbb{R}^{n\times n}$ be symmetric matrices with simple and distinct eigenvalues. Then the "eigenvalue configuration" C is the relative locations of the eigenvalues.

Example $(m = n = 2)$

Let $F = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} 3 & 2 \\ 2 & 9 \end{bmatrix}$

Then $C =$ 

Let $F = \begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix}$ and $G = \begin{bmatrix} -1 & 4 \\ 4 & 3 \end{bmatrix}$

Then $C =$ 

Problem (Conditions for eigenvalue configuration)
In : C, eigenvalue configuration
Out: Condition on F and G for C

Example $(m = n = 2)$

In : $C =$ 

Out: $F = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ and $G = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$

such that

$$\operatorname{sign}(D) = \begin{bmatrix} + & - & + \\ + & - & + \\ + & - & 0 \\ 0 & 0 & + \end{bmatrix} \vee \begin{bmatrix} + & - & + \\ 0 & 0 & + \\ - & + & + \\ + & - & + \end{bmatrix} \vee \cdots$$

In : $C =$ 

Out: $F = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ and $G = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$

such that

$$\operatorname{sign}(D) = \begin{bmatrix} + & - & + \\ + & - & + \\ + & - & 0 \\ 0 & 0 & + \end{bmatrix} \vee \begin{bmatrix} + & - & + \\ + & + & - \\ - & + & + \\ + & - & - \end{bmatrix} \vee \cdots$$

where

$$D = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 2(a_{11} + a_{22} - b_{11} - b_{22}) & (a_{11} + a_{22})^2 - 2(a_{11} + a_{22})(b_{11} + b_{22}) + 4(b_{11}b_{22} - b_{12}^2) \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

We call

- the sign matrices : **sign-configuration transform**,
- the polynomial matrix D: **configuration discriminant**.

Motivation

- Generalization of Descartes' rule of sign: from $F = [0]$ to arbitrary F.
- Applications to many problems in science and engieering.
- Fundamental importance: specialized quantifier elimination problem.

Result (signature-based)

Theorem 1 (Signature-based)

Sign-configuration transform

We have

$$\operatorname{sign}(D) = \bigvee_{S\,:\,C = VH^{-1}\sigma(S)}$$

where

$$V = \left[\delta_{v(s,+),m-t}\right]_{t,s}$$

$$H = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{\otimes m}$$

$$\sigma(S) = \text{signature of } S$$

Configuration discriminant

We have

$$D = \begin{bmatrix} \text{coeffs}\, h_{0\ldots0} \\ \text{coeffs}\, h_{0\ldots1} \\ \vdots \\ \text{coeffs}\, h_{1\ldots1} \end{bmatrix}$$

where

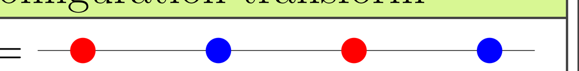
$$h_e = \operatorname{charpoly}\left(f_e(G)\right)$$

$$f_e = f^{(0)^{e_0}} \cdots f^{(m-1)^{e_{m-1}}}$$

$$f = \operatorname{charpoly}(F)$$

Example $(m = n = 2)$

Sign-configuration transform

$C =$ 

$\Uparrow$ encoding

$$C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Uparrow$ $C = VH^{-1}\sigma(S)$

$$\sigma(S) = \begin{bmatrix} 4 & 4 & 1 & 2 \end{bmatrix}^T$$

$\Uparrow$ signature

$$S = \begin{bmatrix} + & - & + \\ + & - & + \\ + & - & 0 \\ 0 & 0 & + \end{bmatrix}$$

Configuration discriminant

$$f = (x - \alpha_{11})(x - \alpha_{22}) + a_{12}^2$$

$$\Downarrow$$

$$f_{01} = f^{(0)^0} f^{(1)^1} = 2x - a_{11} - a_{22}$$

$$\Downarrow$$

$$f_{01}(G) = \begin{bmatrix} -a_{22} - a_{11} + 2b_{11} & 2b_{1,2} \\ 2b_{12} & -a_{22} - a_{11} + 2b_{22} \end{bmatrix}$$

$$\Downarrow$$

$$\text{coeffs}\, h_{01} = \begin{bmatrix} 1 \\ 2(a_{11} + a_{22} - b_{11} - b_{22}) \\ (a_{11} + a_{22})^2 - 2(a_{11} + a_{22})(b_{11} + b_{22}) + 4(b_{11}b_{22} - b_{12}^2)) \end{bmatrix}^T$$

Result (symmetric functions-based)

Theorem 2 (Symmetric functions-based)

Sign-configuration transform

We have

$$\operatorname{sign}(D) = \bigvee_{S\,:\,C = T_m^{-1}v(S)}$$

where

$$(T_m)_{rs} = (-2)^{r-1}\binom{m-s}{r-s}\binom{s}{r}$$

Configuration discriminant

We have

$$D = \begin{bmatrix} \text{coeffs}\, p_1 \\ \vdots \\ \text{coeffs}\, p_m \end{bmatrix}$$

where

$$p_r(a_{ij}, b_{ij}) = \prod_{(i,j)\in Y_r} \left(-x - \prod_{p=1}^{r} \left(\alpha_{i_p} - \beta_j\right)\right)$$

$$Y_r = \{(i,j) : i_1 < \cdots < i_r\}$$

Comparison

	Signature	Symmetric	Descartes
Size of D	$(2^m - 1)\cdot n$	$(2^m - 1)\cdot n$	n

- The size of the configuration discriminant of each approach is the same: $(2^m - 1)\cdot n$
- The above two methods are asymmetric with regard to the sizes of F and G
  - If size of F is fixed, size of discriminant is linear in n
  - In practice, let F be the smaller matrix
- Descartes' rule of signs is for the special case
  $$F = [0] \in \mathbb{R}^{1\times1} \text{ and } G \in \mathbb{R}^{n\times n}.$$
  Setting $m = 1$, the size of the discriminant D for both approaches is $(2^1 - 1)\cdot n = n$

Future Work

- The outputs of Theorems 1 and 2 grow quickly with m and n. Try to prune the outputs (by removing redundancies, logical inconsistencies, etc.)
- Extend to more than 2 real symmetric matrices
- Extend to matrices with arbitrary eigenvalues

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